Persistent random walk on a site-disordered one-dimensional lattice: Photon subdiffusion

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We study the persistent random walk of photons on a one-dimensional lattice of random transmittances. Transmittances at different sites are assumed independent, distributed according to a given probability density $f(t)$. Depending on the behavior of $f(t)$ near $t=0$, diffusive and subdiffusive transports are predicted by the disorder expansion of the mean square-displacement and the effective medium approximation. Monte Carlo simulations confirm the anomalous diffusion of photons. To observe photon subdiffusion experimentally, we suggest a dielectric film stack for realization of a distribution $f(t)$.

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I. INTRODUCTION

Random walks figure prominently in a multitude of different physical problems. This is exemplified by such diverse fields as the polymer physics [1], crystallographic statistics [2,3], transport in disordered media [3–5], bacterial motion, and other types of biological migration [6].

Random walks with *correlated* displacements model a host of phenomena. For example, the vacancy mechanism of atom diffusion in solids incorporates a correlation effect, since an atom has a larger probability to move backward to the hole it just vacated rather than onward [7]. Correlations also arise in diffusion of guest molecules in zeolite channels [8], electron hopping in Coulomb glass [9], motion of excitons at low temperatures in mixed naphthalene crystals [10], etc. Among the correlated walks, the persistent random walk is possibly the simplest one to incorporate a form of momentum in addition to random motion [3,4]. In its basic realization on a one-dimensional lattice, a persistent random walker possesses constant probabilities for either taking a step in the same direction as the immediately preceding one or for reversing its motion. First introduced by Fürth as a model for diffusion in a number of biological problems [11], and shortly after by Taylor in the analysis of turbulent diffusion [12], the persistent random walk model is now generalized to study, e.g., polymers [13], chemotaxis [14], cell movement [15], and general transport mechanisms [16,17].

Recently, the persistent random walk model is utilized in the description of diffusive light transport in foams [18–22] which is well established by experiments [23]. A relatively dry foam consists of cells separated by thin liquid films [24]. Cells in a foam are much larger than the wavelength of light, thus one can employ ray optics and follow a light beam or photon as it is transmitted through the liquid films with a probability t called the intensity transmittance. This naturally leads to a persistent random walk of the photons. Special attention is paid to the light transport in the ordered honeycomb and Kelvin structures, which have been used for an analytic access to the physical properties of disordered foams as exemplified by work on their rheological behavior [25]. Symmetries of hexagonal and tetrakaidecahedral cells allow identifying specific one- and two-dimensional random walks, which are absent in disordered foams [18,21]. Moreover, in a first model, it is assumed that intensity transmittance does not depend on the incidence angle of photons and film thickness. Nevertheless, analytical treatment of these peculiar and simple random walks facilitates interpretation of Monte Carlo simulations, in which topological disorder of foams and exact thin-film intensity transmittance are taken into account [19,21].

In the ordered honeycomb (Kelvin) structure, the one-dimensional persistent walk arises when the photons move perpendicular to a cell edge (face). Thin-film transmittance depends on the film thickness. Films are not expected to have the same thickness. These observations motivates us to consider persistent random walk on a one-dimensional lattice of *random* transmittances. We assume that transmittances at different sites are independent random variables, distributed according to a given probability density $f(t)$.

Our first approach to the problem is based upon a disorder expansion of the mean square-displacement due to Kundu, Parris, and Phillips [26]. Assuming that $\langle 1/t\rangle=\int_0^1 f(t)/t\,dt$ is finite, we validate the classical persistent random walk with an effective transmittance t_{eff}, where $1/t_{eff}=\langle 1/t\rangle$. Inspired by this result, we generalize the effective medium approximation (EMA) formulated by Sahimi, Hughes, Scriven, and Davis [27,28] to investigate the transport on a line with infinite $\langle 1/t\rangle$. We show that if $f(t)\to f(0)$ as $t\to 0$, the mean square-displacement after n steps is proportional to $n/\ln(n)$. If $f(t)\sim f_\alpha t^{-\alpha}$ $(0<\alpha<1)$ as $t\to 0$, we find that the mean square-displacement is proportional to $n^{(2-2\alpha)/(2-\alpha)}$. Our Monte Carlo simulations confirm the anomalous diffusion of photons. Quite interesting, we find that anomalous diffusion of persistent walkers and *hopping* particles on a site-disordered lattice [27–29] are similar. Finally, for the experimental observation of the photon subdiffusion, we suggest a dielectric film stack to realize small transmittances.

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Our paper is organized as follows. In Sec. II we introduce the model. The perturbative approach and the effective medium approximation are discussed in Secs. III and IV, respectively. The numerical treatment and its results are reported in Sec. V. We close with a discussion of our results and conclusions in Sec. VI.

II. MODEL

We consider a one-dimensional lattice random walk in which steps are permitted to the nearest neighbor sites only. We normalize the length and duration of a step to 1. At each site $\boldsymbol{j}$, a walker either takes a step in the same direction as the immediately preceding one with a probability t_j, or reverses its motion with a probability $r_j = 1 - t_j$. Here we assume symmetric transmittances, i.e., $t_{j\to j+1} = t_{j\to j-1} = t_j$, as the transmittance of a thin-film is the same whether the light ray is going to the right $(+)$ or to the left $(-)$ direction.

We assume that (i) transmittance at each site is a random variable, (ii) transmittances at two different sites are independent, (iii) transmittances at all sites are distributed according to a given normalized probability density $f(t)$. Apparently $\int_0^1 f(t)dt = 1$. For any function $h(t)$, we define $\langle h(t)\rangle = \int_0^1 h(t)f(t)dt$.

We denote by $P^+(n,\boldsymbol{j})(P^-(n,\boldsymbol{j}))$ the probability that the walker after its nth step arrives at site $\boldsymbol{j}$ with positive (negative) momentum. A set of two master equations can be established to couple the probabilities at step $n+1$ to the probabilities at step n:

$$P^+(n+1,\boldsymbol{j}) = t_{j-1}P^+(n,\boldsymbol{j}-1) + r_{j-1}P^-(n,\boldsymbol{j}-1),$$

$$P^-(n+1,\boldsymbol{j}) = r_{j+1}P^+(n,\boldsymbol{j}+1) + t_{j+1}P^-(n,\boldsymbol{j}+1). \tag{1}$$

For the description of the photon distribution on the line, we do not need to specify the internal state $(\pm)$ explicitly. That means we are mainly interested in the probability that the photon arrives at position $\boldsymbol{j}$ at step n,

$$P(n,\boldsymbol{j}) = P^+(n,\boldsymbol{j}) + P^-(n,\boldsymbol{j}), \tag{2}$$

from which we extract the first and second moments after n steps as the characteristic features of a random walk:

$$\langle\langle \boldsymbol{j}\rangle\rangle_n = \left\langle \sum_j \boldsymbol{j} P(n,\boldsymbol{j})\right\rangle,$$

$$\langle\langle \boldsymbol{j}^2\rangle\rangle_n = \left\langle \sum_j \boldsymbol{j}^2 P(n,\boldsymbol{j})\right\rangle. \tag{3}$$

Here the first bracket represents an ensemble average over all random transmittances, and the second bracket signifies an average with respect to the distribution $P(n,\boldsymbol{j})$.

One obtains the classical persistent random walk assuming a constant transmittance t at each site. Translational invariance of the medium is then invoked to deduce the exact solution of $P(n,\boldsymbol{j})$ in the framework of characteristic functions (the spatial Fourier transforms of probability distributions) [3,4]. Furthermore, the mean square-displacement of photons after $n\to\infty$ steps can be obtained as

$$\langle \boldsymbol{j}^2\rangle_n = \frac{t}{1-t}n. \tag{4}$$

Considering a lattice of random transmittances with an almost narrow distribution $f(t)$, one may intuitively expect normal diffusion of the photons, and the validity of Eq. (4) with an effective transmittance $\langle t\rangle$. However, a closer inspection reveals that even a few sites with small transmittances (large reflectances) may drastically hinder the photon diffusion: In the extreme limit where two transmittances are zero, photons are confined between two sites. This peculiar aspect of diffusion on a *one*-dimensional lattice rules out the above guess. In the following section, we present a sound perturbative approach to the problem.

III. SYSTEMATIC DISORDER EXPANSION OF MEAN SQUARE-DISPLACEMENT OF THE PHOTONS

Many of the approaches to the transport in disordered media have the disadvantage of being restricted to one-dimensional problems. Here we adopt the method of Kundu *et al.* [26], which is applicable to two- and three-dimensional media. We obtain a series of approximate solutions for photon transport on a disordered line by transforming the master equation (1) to an equivalent but more appropriate integral equation for the characteristic function

$$P^\pm(n,\boldsymbol{\theta}) = \sum_{j=-\infty}^{\infty} P^\pm(n,\boldsymbol{j})e^{i\boldsymbol{j}\boldsymbol{\theta}}. \tag{5}$$

This crucial step can be achieved utilizing the generalized generating functions

$$\widetilde{\mathbf{P}}^\pm(n,\boldsymbol{\theta}) = \sum_j t_j P^\pm(n,\boldsymbol{j})e^{i\boldsymbol{j}\boldsymbol{\theta}}, \tag{6}$$

as will be shown in the following.

First we simplify the set of coupled linear difference equations (1) using the method of the z-transform [3,30] explained in Appendix A:

$$\frac{P^+(z,\boldsymbol{j})}{z} - \frac{P^+(n=0,\boldsymbol{j})}{z} = t_{j-1}P^+(z,\boldsymbol{j}-1) + r_{j-1}P^-(z,\boldsymbol{j}-1),$$

$$\frac{P^-(z,\boldsymbol{j})}{z} - \frac{P^-(n=0,\boldsymbol{j})}{z} = r_{j+1}P^+(z,\boldsymbol{j}+1) + t_{j+1}P^-(z,\boldsymbol{j}+1). \tag{7}$$

We assume the initial conditions $P^+(n=0,\boldsymbol{j}) = P^-(n=0,\boldsymbol{j}) = \delta_{j,0}/2$. Now we multiply both sides of (7) with $\exp(i\boldsymbol{j}\boldsymbol{\theta})$ and sum over j, which leads to

$$\mathbf{M}_3(z,\boldsymbol{\theta})\begin{pmatrix} P^+(z,\boldsymbol{\theta}) \\ P^-(z,\boldsymbol{\theta})\end{pmatrix} = \mathbf{M}_2(\boldsymbol{\theta})\begin{pmatrix} \widetilde{\mathbf{P}}^+(z,\boldsymbol{\theta}) \\ \widetilde{\mathbf{P}}^-(z,\boldsymbol{\theta})\end{pmatrix} + \mathbf{M}_1(z), \tag{8}$$

where

$$\mathbf{M}_3(z,\boldsymbol{\theta}) = \begin{pmatrix} \frac{1}{z} & -e^{i\boldsymbol{\theta}} \\ -e^{-i\boldsymbol{\theta}} & \frac{1}{z}\end{pmatrix},$$

$$\mathbf{M}_2(\boldsymbol{\theta}) = \begin{pmatrix} e^{i\boldsymbol{\theta}} & -e^{i\boldsymbol{\theta}} \\ -e^{-i\boldsymbol{\theta}} & e^{-i\boldsymbol{\theta}} \end{pmatrix},$$

$$\mathbf{M}_1(z) = \begin{pmatrix} \frac{1}{2z} \\ \frac{1}{2z} \end{pmatrix}. \tag{9}$$

Thus $P^{\pm}(z,\boldsymbol{\theta})$ can be easily expressed in terms of $\widetilde{\mathbf{P}}^{\pm}(z,\boldsymbol{\theta})$. From Eqs. (5) and (6)

$$P^{\pm}(z,\boldsymbol{j}) = \frac{1}{2\pi t_j}\int_{-\pi}^{\pi} \widetilde{\mathbf{P}}^{\pm}(z,\phi)e^{-ij\phi}d\phi,$$

$$P^{\pm}(z,\boldsymbol{\theta}) = \sum_j \frac{1}{2\pi t_j}\int_{-\pi}^{\pi} \widetilde{\mathbf{P}}^{\pm}(z,\phi)e^{-ij(\phi-\boldsymbol{\theta})}d\phi, \tag{10}$$

hence we can immediately transform Eq. (8) to an equivalent integral equation for $\widetilde{\mathbf{P}}^{\pm}(z,\boldsymbol{\theta})$. It reads

$$\left(\left\langle \frac{1}{t}\right\rangle \mathbf{I} - \mathbf{M}_3^{-1}(z,\boldsymbol{\theta})\mathbf{M}_2(\boldsymbol{\theta})\right)\begin{pmatrix} \widetilde{\mathbf{P}}^{+}(z,\boldsymbol{\theta}) \\ \widetilde{\mathbf{P}}^{-}(z,\boldsymbol{\theta}) \end{pmatrix} = \mathbf{M}_3^{-1}(z,\boldsymbol{\theta})\mathbf{M}_1(z) - \sum_j \frac{\Delta_j}{2\pi}\int_{-\pi}^{\pi} e^{-ij(\phi-\boldsymbol{\theta})}\begin{pmatrix} \widetilde{\mathbf{P}}^{+}(z,\phi) \\ \widetilde{\mathbf{P}}^{-}(z,\phi) \end{pmatrix} d\phi, \tag{11}$$

where the random variable Δ_j is defined as

$$\Delta_j = \frac{1}{t_j} - \left\langle \frac{1}{t}\right\rangle, \tag{12}$$

and $\mathbf{I}$ is the identity matrix. Note that two assumptions, that none of the transmittances is zero, and $\langle 1/t\rangle$ is finite, are required for the validity of Eqs. (10)–(12).

Successive approximations to the solution of integral equation (11) can be generated by the iteration method. Then ensemble averaging over the random transmittances is straightforward. We use the identity $\langle \Delta_j\rangle = 0$ and the assumption

$$\langle \Delta_j \Delta_{j'}\rangle = \delta_{j,j'}\Delta^2, \tag{13}$$

to obtain the leading terms:

$$\left\langle \begin{pmatrix} \widetilde{\mathbf{P}}^{+}(z,\boldsymbol{\theta}) \\ \widetilde{\mathbf{P}}^{-}(z,\boldsymbol{\theta}) \end{pmatrix} \right\rangle = (1+\Delta^2\mathbf{M}_4(z,\boldsymbol{\theta}))\begin{pmatrix} \widetilde{\mathbf{P}}_0(z,\boldsymbol{\theta}) \\ \widetilde{\mathbf{P}}_0^{*}(z,\boldsymbol{\theta}) \end{pmatrix}, \tag{14}$$

where matrix $\mathbf{M}_4(z,\boldsymbol{\theta})$ is given in Appendix B,

$$\widetilde{\mathbf{P}}_0(z,\boldsymbol{\theta}) = \frac{1}{2\left\langle \frac{1}{t}\right\rangle} \frac{z\left(\left\langle \frac{1}{t}\right\rangle e^{i\boldsymbol{\theta}} - 2\cos\boldsymbol{\theta}\right) + \left\langle \frac{1}{t}\right\rangle}{z^2\left(2 - \left\langle \frac{1}{t}\right\rangle\right) - 2z\cos\boldsymbol{\theta} + \left\langle \frac{1}{t}\right\rangle}, \tag{15}$$

and $*$ denotes the complex conjugation.

The task is now calculating $\langle P^{\pm}(z,\boldsymbol{\theta})\rangle$ from Eq. (8), and using the identity

$$\sum_j \boldsymbol{j}^l P^{\pm}(z,\boldsymbol{j}) = \frac{1}{i^l}\left.\frac{\partial^l P^{\pm}(z,\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^l}\right|_{\boldsymbol{\theta}=0} \quad (l=1,2,\ldots), \tag{16}$$

to obtain mean square-displacements of the photons. To this end,

$$\sum \langle\langle \boldsymbol{j}^2\rangle\rangle_n z^n = \frac{z - 4iz\left.\left(\left\langle \frac{\partial \widetilde{\mathbf{P}}^{+}(z,\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} - \frac{\partial \widetilde{\mathbf{P}}^{-}(z,\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right\rangle\right)\right|_{\boldsymbol{\theta}=0}}{1-z^2},$$

$$\sim \frac{1}{\left\langle \frac{1}{t}\right\rangle - 1}\frac{1}{(1-z)^2} + \frac{\sqrt{2}\Delta^2}{2\left(\left\langle \frac{1}{t}\right\rangle - 1\right)^{2.5}}\frac{1}{(1-z)^{1.5}}, \tag{17}$$

as $z\to 1$. We use Tauberian theorem (A2) to conclude

$$\langle\langle \boldsymbol{j}^2\rangle\rangle_n = \frac{1}{\left\langle \frac{1}{t}\right\rangle - 1}n + \frac{\sqrt{2}\Delta^2}{\sqrt{\pi}\left(\left\langle \frac{1}{t}\right\rangle - 1\right)^{2.5}}\sqrt{n}. \tag{18}$$

Mean square-displacement of the photons after $n\to\infty$ steps is indeed proportional to n. We validate the classical persistent random walk result (4) with an effective transmittance t_{eff}, where

$$\frac{1}{t_{eff}} = \left\langle \frac{1}{t}\right\rangle. \tag{19}$$

IV. EFFECTIVE MEDIUM APPROXIMATION

Stochastic transport in random media is often subdiffusive or superdiffusive [3–5,28]. At this stage, we pay attention to persistent random walk on a line with infinite $\langle 1/t\rangle$, where the normal diffusion is not guaranteed. Our approach to the problem is based upon a variant of effective medium approximation (EMA) developed in Refs. [27,31].

To facilitate solution of Eq. (7) we introduce a reference lattice or average medium, with all intensity transmittances (reflectances) equal to $t_e(z)$ ($r_e(z)$), and probabilities $P_e^{\pm}(z,\boldsymbol{j})$, so that

$$\frac{P_e^{+}(z,\boldsymbol{j})}{z} - \frac{P^{+}(n=0,\boldsymbol{j})}{z} = t_e(z)P_e^{+}(z,\boldsymbol{j}-1) + r_e(z)P_e^{-}(z,\boldsymbol{j}-1),$$

$$\frac{P_e^{-}(z,\boldsymbol{j})}{z} - \frac{P^{-}(n=0,\boldsymbol{j})}{z} = r_e(z)P_e^{+}(z,\boldsymbol{j}+1) + t_e(z)P_e^{-}(z,\boldsymbol{j}+1). \tag{20}$$

EMA determines $t_e(z)$ and $r_e(z)$ in a self-consistent manner, in which the role of distribution $f(t)$ is manifest. This is done by taking a cluster of random transmittances from the original distribution, and embedding it into the effective medium. We then require that *average* of site occupation probabilities of the decorated medium duplicate $P_e^{\pm}(z,\boldsymbol{j})$ of the effective medium. We will sketch the method in the following.

Subtracting Eqs. (7) and (20), we obtain

$$\frac{1}{z}\begin{pmatrix} Q^+(z,\boldsymbol{j}) \\ Q^-(z,\boldsymbol{j}) \end{pmatrix} - \mathbf{T}^-(z)\begin{pmatrix} Q^+(z,\boldsymbol{j}-1) \\ Q^-(z,\boldsymbol{j}-1) \end{pmatrix} - \mathbf{T}^+(z)\begin{pmatrix} Q^+(z,\boldsymbol{j}+1) \\ Q^-(z,\boldsymbol{j}+1) \end{pmatrix}$$
$$= \left[\begin{pmatrix} t_{\boldsymbol{j}-1} & r_{\boldsymbol{j}-1} \\ 0 & 0 \end{pmatrix} - \mathbf{T}^-(z)\right]\begin{pmatrix} P^+(z,\boldsymbol{j}-1) \\ P^-(z,\boldsymbol{j}-1) \end{pmatrix}$$
$$+ \left[\begin{pmatrix} 0 & 0 \\ r_{\boldsymbol{j}+1} & t_{\boldsymbol{j}+1} \end{pmatrix} - \mathbf{T}^+(z)\right]\begin{pmatrix} P^+(z,\boldsymbol{j}+1) \\ P^-(z,\boldsymbol{j}+1) \end{pmatrix}, \quad (21)$$

where

$$\begin{pmatrix} Q^+(z,\boldsymbol{j}) \\ Q^-(z,\boldsymbol{j}) \end{pmatrix} = \begin{pmatrix} P^+(z,\boldsymbol{j}) \\ P^-(z,\boldsymbol{j}) \end{pmatrix} - \begin{pmatrix} P_e^+(z,\boldsymbol{j}) \\ P_e^-(z,\boldsymbol{j}) \end{pmatrix},$$

$$\mathbf{T}^-(z) = \begin{pmatrix} t_e(z) & r_e(z) \\ 0 & 0 \end{pmatrix},$$

$$\mathbf{T}^+(z) = \begin{pmatrix} 0 & 0 \\ r_e(z) & t_e(z) \end{pmatrix}. \quad (22)$$

Equation (21) suggests to define an associated Green function

$$\mathbf{G}(z,\boldsymbol{j}) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$$

by the equation

$$\frac{1}{z}\mathbf{G}(z,\boldsymbol{j}) - \mathbf{T}^-(z)\mathbf{G}(z,\boldsymbol{j}-1) - \mathbf{T}^+(z)\mathbf{G}(z,\boldsymbol{j}+1) = \delta_{\boldsymbol{j},0}\mathbf{I}, \quad (23)$$

whose solution is the inverse Fourier transform of

$$\mathbf{G}(z,\boldsymbol{\theta}) = \frac{z^2}{1 - 2zt_e(z)\cos\boldsymbol{\theta} + z^2[t_e^2(z) - r_e^2(z)]} \times \begin{pmatrix} \frac{1}{z} - t_e(z)e^{-i\boldsymbol{\theta}} & r_e(z)e^{i\boldsymbol{\theta}} \\ r_e(z)e^{-i\boldsymbol{\theta}} & \frac{1}{z} - t_e(z)e^{i\boldsymbol{\theta}} \end{pmatrix}. \quad (24)$$

For the present, we consider only the simplest approximation, and embed *one* random transmittance at site $\boldsymbol{l}$ of the effective medium. Then solution of Eq. (21) is

$$\begin{pmatrix} Q^+(z,\boldsymbol{j}) \\ Q^-(z,\boldsymbol{j}) \end{pmatrix} = \int_0^{2\pi} \mathbf{G}(z,\boldsymbol{\theta})\mathbf{S}(z,\boldsymbol{\theta})e^{-i\boldsymbol{\theta}(\boldsymbol{j}-\boldsymbol{l})} \times \begin{pmatrix} P^+(z,\boldsymbol{l}) \\ P^-(z,\boldsymbol{l}) \end{pmatrix}\frac{d\boldsymbol{\theta}}{2\pi}, \quad (25)$$

where

$$\mathbf{S}(z,\boldsymbol{\theta}) = \begin{pmatrix} [t_{\boldsymbol{l}} - t_e(z)]e^{i\boldsymbol{\theta}} & [r_{\boldsymbol{l}} - r_e(z)]e^{i\boldsymbol{\theta}} \\ [r_{\boldsymbol{l}} - r_e(z)]e^{-i\boldsymbol{\theta}} & [t_{\boldsymbol{l}} - t_e(z)]e^{-i\boldsymbol{\theta}} \end{pmatrix}. \quad (26)$$

Self-consistency equation is $\langle P^{\pm}(z,\boldsymbol{l})\rangle = P_e^{\pm}(z,\boldsymbol{l})$, or

$$\left\langle \left[\mathbf{I} - \int_0^{2\pi} \mathbf{G}(z,\boldsymbol{\theta})\mathbf{S}(z,\boldsymbol{\theta})\frac{d\boldsymbol{\theta}}{2\pi}\right]^{-1} \right\rangle = \mathbf{I}. \quad (27)$$

The above matrix equation leads to two independent self-consistency conditions. Choosing $t_e(z)+r_e(z)=1$, one of the conditions can be satisfied. It is a signature of the success of effective medium to represent the original random medium, where $t_{\boldsymbol{j}}+r_{\boldsymbol{j}}=1$ holds at any site $\boldsymbol{j}$. The second self-consistency condition then determines $t_e(z)$:

$$\frac{1}{t_e(z)} = \int_0^1 \frac{f(t)dt}{[t_e(z)-t]\frac{\sqrt{1-z^2}}{\sqrt{1-[2t_e(z)-1]^2 z^2}} + t}, \quad (28)$$

in which the role of distribution $f(t)$ is manifest.

Z-transform of mean square-displacement of the photons in the effective medium can be obtained as

$$\sum_{n=0}^{\infty} \langle\langle \boldsymbol{j}^2 \rangle\rangle_n z^n = \frac{z}{(1-z)^2}\frac{1+z[2t_e(z)-1]}{1-z[2t_e(z)-1]}. \quad (29)$$

We are interested in the long time behavior, thus Tauberian theorems suggest to analyze Eqs. (28) and (29) in the limit $z\to 1$.

First we assume that $\langle 1/t\rangle$ is finite and $t_e(z)$ has no singularity in the limit $z\to 1$. Then Eq. (28) yields

$$\frac{1}{t_e(z)} = \int_0^1 \frac{f(t)dt}{t}, \quad (30)$$

in accordance with our second assumption. We deduce from Eqs. (29) and (A2) that

$$\langle\langle \boldsymbol{j}^2 \rangle\rangle_n = \frac{1}{\left\langle \frac{1}{t} \right\rangle - 1} n, \quad (31)$$

therefore the system evolves diffusively. This is in complete agreement with the predictions of Sec. III.

When $\langle 1/t\rangle$ is infinite, the behavior of $1/t_e(z)$ is determined by the behavior of $f(t)$ at small values of t. Let us assume that $f(t)$ has a finite derivative at $t=0$. We can decompose the integral in Eq. (28) into a sum:

$$\frac{1}{t_e(z)} = \int_0^{\epsilon} \frac{f(0)dt}{[t_e(z)-t]\frac{\sqrt{1-z^2}}{\sqrt{1-[2t_e(z)-1]^2 z^2}} + t} + \int_0^{\epsilon} \frac{[f(t)-f(0)]dt}{[t_e(z)-t]\frac{\sqrt{1-z^2}}{\sqrt{1-[2t_e(z)-1]^2 z^2}} + t} + \int_{\epsilon}^1 \frac{f(t)dt}{[t_e(z)-t]\frac{\sqrt{1-z^2}}{\sqrt{1-[2t_e(z)-1]^2 z^2}} + t}, \quad (32)$$

where ϵ is a small number. By assumption $f(t)-f(0)=tf'(0)$ near $t=0$, and the factor t cancels the potential singularity in the second term as $z\to 1$. Indeed the only singular behavior can come from the first term. Equation (32) then leads to

$$\frac{1}{t_e(z)} \approx -f(0)\ln\left[\frac{t_e(z)\sqrt{1-z^2}}{\sqrt{1-[2t_e(z)-1]^2z^2}}\right]. \tag{33}$$

Methods for the asymptotic solution of transcendental equations [32] are invoked to obtain

$$t_e(z) \approx \frac{-2}{f(0)\ln(1-z)}. \tag{34}$$

We deduce from Eqs. (29) and (A3) that

$$\langle\langle \boldsymbol{j}^2\rangle\rangle_n = \frac{2}{f(0)}\frac{n}{\ln(n)}, \tag{35}$$

thus the transport is subdiffusive.

We now investigate the cases in which $f'(0)$ is infinite. If $f(t)\sim f_\alpha t^{-\alpha}$ $(0<\alpha<1)$ as $t\to 0$, self-consistency equation (28) yields

$$[t_e(z)]^{\alpha-1} = \frac{\pi f_\alpha}{\sin(\pi\alpha)}\left[\frac{1-[2t_e(z)-1]^2z^2}{1-z^2}\right]^{\alpha/2}, \tag{36}$$

where we have used $\int_0^\infty x^{-\alpha}/(1+x)dx=\pi/\sin(\pi\alpha)$. In this case we find

$$t_e(z) \approx \left[2^{-\alpha/2}\frac{\sin(\pi\alpha)}{\pi f_\alpha}\right]^{2/(2-\alpha)}(1-z)^{\alpha/(2-\alpha)}, \tag{37}$$

and

$$\langle\langle \boldsymbol{j}^2\rangle\rangle_n = \frac{\left[2^{-\alpha/2}\frac{\sin(\pi\alpha)}{\pi f_\alpha}\right]^{2/(2-\alpha)}}{\Gamma\left(\frac{4-3\alpha}{2-\alpha}\right)} n^{(2-2\alpha)/(2-\alpha)}. \tag{38}$$

Apparently, $0<(2-2\alpha)/(2-\alpha)<1$ and the transport is subdiffusive.

V. NUMERICAL SIMULATIONS

The predictions of EMA can be inspected by numerical simulations. The computer program produces 400 media, whose transmittances are distributed according to a given $f(t)$. For each medium, it takes 10^3 photons at the initial position $\boldsymbol{j}=0$ and generates the trajectory of each photon following a standard Monte Carlo procedure.

For the binary distribution $f(t)=p_1\delta(t-t_1)+(1-p_1)\delta(t-1+t_1)$, where $p_1\in[0.1,0.2,\ldots,0.9]$ and $t_1\in[0.1,0.2,0.3,0.4]$, statistics of the photon cloud is evaluated at times $n\in[7000,7100,\ldots,10\,000]$. The mean-square displacement $\langle\langle \boldsymbol{j}^2\rangle\rangle_n$ is computed for each snapshot at time n, and then fitted to $Dn+O$ by the method of linear regression. An offset O takes into account the initial ballistic regime. Figure 1 shows the excellent agreement between numerical simulations and $D(p_1,t_1)=t_1(1-t_1)/(p_1-2p_1t_1+t_1^2)$, predicted by Eq. (31). The maximum differences ±0.04 are comparable to the errorbars ±0.01 which linear regression anticipates.

For the uniform distribution $f(t)=1$ $(0\leqslant t\leqslant 1)$, statistics is evaluated at times $n\in[10\,000,25\,000,\ldots,400\,000]$. In Fig. 2, $\langle\langle \boldsymbol{j}^2\rangle\rangle_n/n$, and $2/\ln(n)$ are plotted. This figure confirms the EMA prediction (35).

For $f(t)=(1-\alpha)t^{-\alpha}$ $(0\leqslant t\leqslant 1)$, where $\alpha\in[0.1,0.2,\ldots,0.9]$, statistics is evaluated at times $n\in[70\,000,71\,000,\ldots,100\,000]$. $\ln[\langle\langle \boldsymbol{j}^2\rangle\rangle_n]$ is computed for each snapshot at time n, and then fitted to $\gamma\ln(n)+O$, where O is an offset. In Fig. 3, we plot γ and $(2-2\alpha)/(2-\alpha)$ as a function of α, to inspect the EMA prediction (38). The differences are greater than the errorbars which regression anticipates, but the overall agreement is excellent.

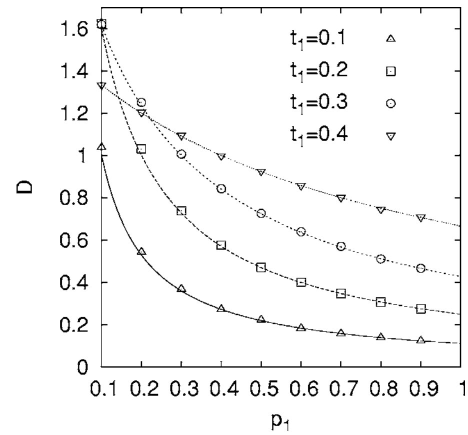

FIG. 1. The diffusion constant as a function of p_1 and t_1 which parametrize $f(t)=p_1\delta(t-t_1)+(1-p_1)\delta(t-1+t_1)$. Theoretical and Monte Carlo simulation results are denoted, respectively, by lines and points.

VI. DISCUSSIONS AND CONCLUSIONS

In the present paper, we address the persistent random walk on a one-dimensional lattice of random transmittances. The photon transport is diffusive, provided that $\langle 1/t\rangle$ is finite (class I). The transmittance of the effective medium is given by Eq. (19). It expresses the fact that a few sites with small transmittances (large reflectances) may hinder the photon diffusion. As percolation properties [28], this feature is in-

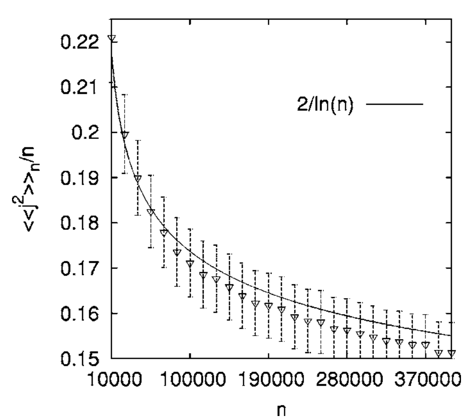

FIG. 2. $\langle\langle \boldsymbol{j}^2\rangle\rangle_n/n$ as function of n, for $f(t)=1$ $(0<t<1)$. Theoretical and simulation results are denoted, respectively, by line and points.

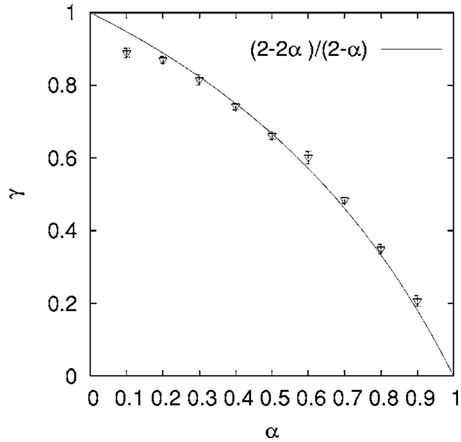

FIG. 3. $\gamma=\ln[\langle\langle \boldsymbol{j}^2\rangle\rangle_n]/\ln[n]$ as $n\to\infty$, vs α which parametrizes $f(t)=(1-\alpha)t^{-\alpha}$. Theoretical and simulation results are denoted, respectively, by line and points.

duced by the dimensionality of the lattice. Furthermore, Eq. (18) shows that fluctuations in transmittances give rise to a $\sqrt{n}$ correction to the mean square-displacement. Nevertheless, this correction is negligible at times $n\gg\Delta^4/(\langle 1/t\rangle-1)^3$.

A photon steps back by each reflection. Intuitively, one expects the abundance of large reflectances to drastically decrease excursion of the photons. Our EMA predicts a subdiffusive transport when $\langle 1/t\rangle$ is infinite. The self-consistency equation (32) divides such distributions to two classes, II and III, which are summarized in Table I. Remarkably, for class III the exponent of mean square-displacement is distribution dependent (nonuniversality).

It would be instructive to compare transport of persistent walkers and excitation dynamics in random one-dimensional systems [27–29]. Hopping conduction is described by the master equation

$$\frac{\partial P(\tau,\boldsymbol{j})}{\partial\tau}=w_{\boldsymbol{j},\boldsymbol{j}-1}[P(\tau,\boldsymbol{j}-1)-P(\tau,\boldsymbol{j})] + w_{\boldsymbol{j},\boldsymbol{j}+1}[P(\tau,\boldsymbol{j}+1)-P(\tau,\boldsymbol{j})], \qquad (39)$$

where $0\leqslant w_{\boldsymbol{j},\boldsymbol{j}+1}=w_{\boldsymbol{j}+1,\boldsymbol{j}}\leqslant\infty$ is the random transition rate between sites $\boldsymbol{j}$ and $\boldsymbol{j}+1$, and τ is the continuous time. Correlated steps are the essence of (1), while independent steps are the base of (39). Despite this celebrated difference, Table I reveals remarkable similarities.

As we mentioned in Sec. I, the persistent walk on a one-dimensional lattice of random transmittances arises when the photons move perpendicular to an edge (a face) of the honeycomb (Kelvin) structure. At normal incidence on a film of random thickness d_W and refractive index n_W,

$$t=1-\frac{2r_W^2(1-\cos\beta_W)}{r_W^4-2r_W^2\cos\beta_W+1}, \qquad (40)$$

where $r_W=(n_W-1)/(n_W+1)$, $\beta_W=4\pi d_W n_W/\lambda$, and λ is the light wavelenght [19,21]. For foams $n_W\sim 1.34$ (water) and $r_W\neq\pm 1$, hence the above transmittance never approaches zero, $\langle 1/t\rangle$ exists, and the photon transport is diffusive.

In a realization of a distribution $f(t)\sim f_\alpha t^{-\alpha}$ as $t\to 0$, small transmittances prevail. For an experimental observation of the photon subdiffusion, we propose dielectric mirrors to obtain desired small transmittances. Consider the multilayer cofiguration

$$\mathrm{HL\ HL}\cdots\overbrace{\mathrm{HL}}^{N}\ \mathrm{W\ LH}\cdots\mathrm{LH}\ \overbrace{\mathrm{LH}}^{N},$$

where H (L) is a quarter-wave layer with high (low) index n_H (n_L), and N is the number of HL (LH) pairs. W is a layer of thickness d_W and index n_W. Following the theory of multilayer films [33], we obtain

$$t=1-\frac{(r'_W-1)^2(1-\cos\beta_W)}{r'^2_W(1-\cos\beta_W)+2r'_W(3+\cos\beta_W)+1-\cos\beta_W}, \qquad (41)$$

where $r'_W=n_W^2(n_H/n_L)^{4N}$. Apparently, as N increases the transmittance rapidly approaches zero. For example, suppose $n_H=2.40$ (TiO_2), $n_L=1.38$ (MgF_2), $n_W=1.46$ (SiO_2), $\lambda\sim 550$ nm, and $\beta_W=\pi$ (quarter-wave layer). For 1, 2, and 5 pairs, the transmittances are 0.185 604, 0.022 174, and 0.000 029, respectively. In this realization of small transmittances, two points are considered. First, transmittance of the multilayer is the same whether light rays go to the right or the left direction. Second, parameters of the multilayer, es-

TABLE I. EMA prediction of ultimate growth with time of the mean square-displacement.

	Photons		Excitons[a]	
	$f(t)$	$\langle\langle \boldsymbol{j}^2\rangle\rangle_n$	$f(w)$	$\langle\langle \boldsymbol{j}^2\rangle\rangle_\tau$
I	$\left\langle \frac{1}{t}\right\rangle\neq\infty$	n	$\left\langle \frac{1}{w}\right\rangle\neq\infty$	τ
II	$\left\langle \frac{1}{t}\right\rangle=\infty, f(t)\to f(0)$ as $t\to 0$	$n/\ln(n)$	$\left\langle \frac{1}{w}\right\rangle=\infty, f(w)\to f(0)$ as $w\to 0$	$\tau/\ln\tau$
III	$f(t)\sim t^{-\alpha}$ as $t\to 0$	$n^{(2-2\alpha)/(2-\alpha)}$	$f(w)\sim w^{-\alpha}$ as $w\to 0$	$\tau^{(2-2\alpha)/(2-\alpha)}$

[a]From Ref. [27]. τ and w denote the continuous time and the transition rate between the nearest neighbors, respectively.

pecially N and d_W, can be varied to fine tune the transmittance. Moreover, absorption of the multilayer is negligible. Note that thin metalic slabs are highly reflecting, but absorptive.

Our studies can be extended to higher dimensional lattices, and special two-dimensional photon paths in the honeycomb and Kelvin structures [18,21]. Another path to pursue is the creation of artificial one-dimensional structures to observe anomalous diffusion of photons.

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APPENDIX A: z-TRANSFORM

The z-transform $F(z)$ of a function $F(n)$ of a discrete variable $n=0,1,2,\ldots$ is defined by

$$F(z)=\sum_{n=0}^{\infty}F(n)z^n. \tag{A1}$$

One then derives the z-transform of $F(n+1)$ simply as $F(z)/z-F(n=0)/z$. Note the similarities of this rule with the Laplace transform of the time derivative of a continuous function [3,30].

Under specified conditions the singular behavior of $F(z)$ can be used to determine the asymptotic behavior of $F(n)$ for large n (Tauberian theorems) [3]. For example:

$$F(z)\sim\frac{\Gamma(1-\alpha)}{(1-z)^{1-\alpha}}\to F(n)\sim\frac{1}{n^{\alpha}}, \tag{A2}$$

$$F(z)\sim\frac{1}{\ln\left(\frac{1}{1-z}\right)(1-z)^{\alpha}}\to F(n)\sim\frac{n^{\alpha-1}}{\Gamma(\alpha)\ln(n)}, \tag{A3}$$

where $\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt$.

APPENDIX B: MATRIX $\mathbf{M}_4(z,\boldsymbol{\theta})$

The matrix $\mathbf{M}_4(z,\boldsymbol{\theta})$ introduced in Eq. (14) can be conveniently written as

$$\mathbf{M}_4(z,\boldsymbol{\theta})=\begin{pmatrix} a(z,\boldsymbol{\theta})+b(z,\boldsymbol{\theta}) & a(z,\boldsymbol{\theta})-b(z,\boldsymbol{\theta}) \\ a(z,\boldsymbol{\theta})-b^*(z,\boldsymbol{\theta}) & a(z,\boldsymbol{\theta})+b^*(z,\boldsymbol{\theta}) \end{pmatrix}, \tag{B1}$$

where

$$a(z,\boldsymbol{\theta})=\frac{1}{2\left\langle\frac{1}{t}\right\rangle^2}, \tag{B2}$$

$$b(z,\boldsymbol{\theta})=\frac{-1}{2\left\langle\frac{1}{t}\right\rangle}\sqrt{\frac{1-z^2}{\left\langle\frac{1}{t}\right\rangle^2-z^2\left(\left\langle\frac{1}{t}\right\rangle-2\right)^2}}\times\frac{z^2\left\langle\frac{1}{t}\right\rangle-2iz\sin\boldsymbol{\theta}-\left\langle\frac{1}{t}\right\rangle}{z^2\left(2-\left\langle\frac{1}{t}\right\rangle\right)-2z\cos\boldsymbol{\theta}+\left\langle\frac{1}{t}\right\rangle}.$$

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